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ON AN EQUILIBRIUM STATE IN TWO-SECTOR MODEL OF ECONOMIC DYNAMICS *

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Abstract. The model of economic dynamics, consisting of two units that produce, respectively, means of production and objects of commodities. We investigate the equilibrium state of the model. The technique of superlinear mappings is used. The types of equilibrium model are defined.

Ключевые слова: Equilibrium, effective trajectory, superlinear mappings, production function.

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1. Introduction

Consider the two-product model Z_t of the economic dynamics. The first unit produces the means of production, and the other objects of commodities. The vector $x = (x^1, x^2) \in \left(R_+^2\right)^2$ is a model state: here $x^i = \left(K^i, L^i\right) \in R_+^2$; K^i are the main funds, L^i - labor force in the i-th division $(i = 1,2)$. Production activity of the i-th sector at the time t is described by the production function $F_t^i : R_+^2 \to R_+$ and safety coefficient $0 \le v_t^i < 1$ $(i = 1,2)$ of the funds [1,2,6].

The rate of the salary which coincides in the first and in the second divisions is assumed known. Switching from state $x_t = \left(K_t^1, L_t^1, K_t^2, L_t^2\right)$ to state $x_{t+1} = \left(K_{t+1}^1, L_{t+1}^1, K_{t+1}^2, L_{t+1}^2\right)$ is possible if

$$K_{t+1}^1 + K_{t+1}^2 \le v_t^1 K_t^1 + v_t^2 K_t^2 + F_t^1\left(K_t^1, L_t^1\right), \tag{1}$$

$$\omega_{t+i}\left(L_{t+1}^1 + L_{t+1}^2\right) \le F_t^2\left(K_t^2, L_t^2\right), \tag{2}$$

$$F_t^1\left(K_t^1, L_t^1\right) = \min\left(\frac{K_t^1}{C_t^{11}}, \frac{L_t^1}{C_t^{21}}\right), \tag{3}$$

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$$F_t^2\left(K_t^2, L_t^2\right) = \min\left(\frac{K_t^2}{C_t^{12}}, \frac{L_t^2}{C_t^{22}}\right), \tag{4}$$

$$C_t^{ij} > 0 \qquad (i, j = 1,2)$$

Here the coefficients C_t^{ij} stand for the the product number of the i-th division that is necessary for producing the unit product of the j-th division at the moment t. Denote by a_t productive mapping of the model [3,5]. The set $a_t(x_t)$ consists of the vectors x_{t+1}, for which the relation (1) and (2) are valid.

2. Main results. Investigating the model Z_t we use the simple superlinear mappings [3,5] of the form

$$\sigma(K,L) = \left\{\left(K^1, L^1\right) \middle| K^1 \geq 0, L^1 \geq 0, K^1 + \omega L^1 \leq vK + \min\left(\frac{K}{C^1}, \frac{L}{C^2}\right)\right\}, \tag{5}$$

where $C^1 > 0,\ C^2 > 0$ [2,4].

We write Neumann-Gale model [4] Z, defined by the mapping σ in the form $Z = (\sigma, \omega, C^1, C^2)$.

Consider the function

$$f(\eta) = \min\left(\frac{\eta}{C^1}, \frac{1}{C^2}\right), \quad \eta > 0. \tag{6}$$

Lemma 1. Let

$$g(\eta) = \frac{v\eta + f(\eta)}{\eta + \omega}, \eta > 0, \tag{7}$$

where v, ω are some constants. Then if $\omega v < \frac{1}{C^2}$, the function g attains its minimum in the interval $(0,+\infty)$, moreover at the only point $\overline{\eta} = \frac{C^1}{C^2}$. If $\omega v \geq \frac{1}{C^2}$, then the function g strictly increases in this interval.

Proof. Using (2) and (3) we get

$$\max_{\eta>0} g(\eta) = \max_{\eta>0} \min\left(\frac{\eta\left(v + \frac{1}{C^1}\right)}{\eta + \omega}, \frac{v\eta + \frac{1}{C^2}}{\eta + \omega}\right). \tag{8}$$

Consider the function

$$Q(\eta)=\frac{\eta\left(v+\frac{1}{C^1}\right)}{\eta+\omega},\ \ P(\eta)=\frac{v\eta+\frac{1}{C^2}}{\eta+\omega}$$

The function $Q(\eta)$ increases and $Q(0)=0$, $\lim_{\eta\to+\infty} Q(\eta)=v+\frac{1}{C^1}$. The function $P\Phi(\eta)$ by $\omega v<\frac{1}{C^2}$ decreases and $P(0)>0$, $\lim_{\eta\to+\infty} P(\eta)=v$. Moreover both these functions are continuous. From these properties of the functions $\mathrm{Q}(\eta), P(\eta)$ and from (8) it follows that the maximum in (8) is reached at the point $\overline{\eta}$ that indeed a solution of the equation

$$\mathrm{Q}(\eta)=P(\eta).$$

It follows from this that $\overline{\eta}=\frac{\mathrm{C}^1}{\mathrm{C}^2}$. If $\omega v\geq\frac{1}{C^2}$, then $P(\eta)$ and the function $g(\eta)$ increases in the interval $(0,+\infty)$. Lemma is proved.

Take $Z=\left(v,\omega,C^1,C^2\right)$. Here and later on we'll use the denotations

$$\overline{\eta}(Z)=\begin{cases}\frac{C^1}{C^2},\ \ if\ \ \omega v<\frac{1}{\mathrm{C}^2}\\ +\infty,\ \ if\ \ \omega v\geq\frac{1}{C^2}\ \ ,\end{cases} \tag{9}$$

$$\alpha(Z)=\begin{cases}\frac{1+vC^1}{C^1+\omega C^2},\ \ if\ \ \omega\mathrm{v}<\frac{1}{\mathrm{C}^2}\\ v,\ \ \ \ if\ \ \omega v\geq\frac{1}{\mathrm{C}^2}\ \ \ \ \ \ \ \ .\end{cases} \tag{10}$$

It is easy to derive $\alpha(Z)=\sup_{\eta>0} g(\eta)$ (11) from the proof of Lemma 1 where g is a function defined by formula (8).

Note. Later we shall consider the product function of the form $\mathrm{F}(\mathrm{K},\mathrm{L})=b\min\left(\frac{K}{C^1},\frac{L}{C^2}\right)$. Since considering Lemma 1 in the virtue of Lemma1 the number $\overline{\eta}$ depends not on the coefficients C^1, C^2 but their ratio, for the mentioned functions $\overline{\eta}$ does not depend on b.

Proposition 1. Consider the model $\mathrm{Z}=\left(v,\omega,\mathrm{C}^1,\mathrm{C}^2\right)$. Let the point $(\overline{K},\overline{L})$ be such that

$$\max_{\substack{K\geq 0, L\geq 0\\ K+L\neq 0}} \frac{vK+\min\left(\frac{K}{C^1},\frac{L}{C^2}\right)}{K+\omega L} = \frac{v\overline{K}+\min\left(\frac{\overline{K}}{C^1},\frac{\overline{L}}{C^2}\right)}{K+\omega L} \tag{12}$$

and $\overline{\mathrm{K}}+\omega\overline{\mathrm{L}}=1$. Take $\overline{p}=(1,\omega)$. Then the triplet $(\alpha(Z),\overline{x},\overline{p})$ forms the equilibrium state for the model (v,ω,C^1,C^2) and

1) if $wv<\frac{1}{C^2}$, then $\overline{K}=\frac{C^1}{C^1+\omega C^2}$, $\overline{L}=\frac{C^2}{C^1+\omega C^2}$ (13)

2) if $\omega v\geq\frac{1}{C^2}$, then $\overline{K}=1, \overline{L}=0$.

Proof. It is clear that

$$\max_{\substack{K\geq 0, L\geq 0\\ K+L\neq 0}} \frac{vK+\min\left(\frac{K}{C^1},\frac{L}{C^2}\right)}{K+\omega L} = \sup_{\eta>0} g(\eta)=\alpha(Z) \tag{14}$$

By $\omega v<\frac{1}{C^2}$ in the virtue of Lemma 1 supremum in (11) is reached at $\overline{\eta}(Z)=\frac{C^1}{C^2}$. Let the vector $\overline{x}=(\overline{K},\overline{L})$ is such that $\overline{\eta}(Z)=\frac{\overline{K}}{\overline{L}}$, $\overline{K}+\omega\overline{L}=1$; Since $\overline{\eta}(Z)=\frac{C^1}{C^2}$ it is easy to check that $\overline{K},\overline{L}$ are calculated by the formula (13).

If $\omega v\geq\frac{1}{C^2}$ then following Lemma 1 the function $g(\eta)$ increases in the interval $(0,+\infty)$, and from this we get $(\overline{K},\overline{L})=(1,0)$. The triplet $(\alpha(z),\overline{x},\overline{p})$ is an equilibrium state if the following relations are true:

1) $\alpha(Z)\overline{x}\in a(\overline{x})$, where $\overline{x}=(\overline{K},\overline{L})$;

2) $[\overline{p},y]\leq\alpha(Z)[\overline{p},x]$ for $y\in a(x)$;

3) $[\overline{p},\overline{x}]>0$. The last inequality is obvious.

The first relation immediately follows from (5).

Now we show that for all $y\in a(x)$ the inequality $[\overline{p},y]\leq\alpha(Z)[\overline{p},x]$ is valid. Let $x=(K,L)$, $y=(K',L')$. Then from (5) we have

$$[\overline{p},y]=K'+\omega L'\leq vK+\min\left(\frac{K}{C^1},\frac{L}{C^2}\right).$$

Since $\alpha(Z) \geq \dfrac{vK + \min\left(\dfrac{K}{C^1}, \dfrac{L}{C^2}\right)}{K + \omega L}$ for any (K, L), then $[\overline{\mathrm{p}}, \mathrm{y}] \leq vK + \min\left(\dfrac{K}{C^1}, \dfrac{L}{C^2}\right) \leq \alpha(Z)[\overline{p}, x]$.

Proposition is proved.

First let us consider the stationary case of the model $\mathrm{Z_t}$ given above: $v_t^i = v^i,\ \omega_t = \omega,\ C_t^{ij} = C^{ij}$ for all $t = 0,1,2,...;\ i, j = 1,2$. In this case the model Z_t turns to Neumann-Gale model Z [4].

One way to construct a trajectory in the model Z is the use of equilibrium mechanisms [3]. We describe these mechanisms. Assume that the vector of prices $p = \left(p^1, p^2\right)$ is given. Without loss of generality, we assume that the price of the funds is $p^1 = 1$; it is prefer to write p^2 as $p^2 = b\omega$ and then we have $p = (1, b\omega)$, where $b > 0$. Consider the expected combined wealth of the divisions at these prices and vector resources $\mathrm{x} = \left(\mathrm{K}, \mathrm{L}\right)$

$$\begin{aligned} W^1(x, p) &= v^1 K + \min\left(\frac{K}{C^{11}}, \frac{L}{C^{21}}\right), \\ W^2(x, p) &= v^2 K + b\min\left(\frac{K}{C^{12}}, \frac{L}{C^{22}}\right). \end{aligned} \tag{15}$$

Then W^1 and W^2 are considered as utility functions of the production units. Let $\left(K,L\right)$ be a vector distributed resources, λ^1, λ^2 be given budgets of the units. Consider the equilibrium model M with fixed budgets, determined by the quantities $\mathrm{W}^1, W^2, \lambda^1, \lambda^2, (K, L)$.

Let

$$X = \left(K,L\right),\ \chi = \frac{K}{L},\ \chi^1 = \frac{C^{11}}{C^{21}},\ \chi^2 = \frac{C^{12}}{C^{22}},\ u^1 = \frac{1}{v^1 C^{21}}, u^2 = \frac{b}{v^2 C^{22}}. \tag{16}$$

Assume that (q, x^1, x^2) is an equilibrium state of this model, i.e. $x^1 + x^2 = X$ and the vector x^i is a solution of the problem

$$W^i(x, p) \to \max \tag{17}$$

with condition $x \geq 0, \quad [q, x] \leq \lambda^i \quad (i = 1,2)$.

Denote by such a vector $\overline{x}^i$ that $[q, \overline{x}^i] = 1$ and

$$\frac{W^i\left(\overline{x}^i, p\right)}{[q, \overline{x}^i]} = \max_{x \geq 0} \frac{W^i(x, p)}{[q, x]} \tag{18}$$

Then the equilibrium vector x^i has a form

$$x^i = \lambda^i \bar{x}^i \quad (i = 1,2) \tag{19}$$

Let us show the form of an equilibrium state depending on the assets $\chi = \frac{K}{L}$ of the state (K, L) and parameter $\mu = \frac{\lambda^1}{\lambda^2}$. Let $q = (q^1, q^2)$ be equilibrium prices. First of all let us note that $q^1 > 0$. Really, if $q^1 = 0$ then $q^2 > 0$ and the problems (17) turn to

$$\max_{q^2 L \le \lambda^i} W^i(x, p) = +\infty,$$

and this contradicts to the definition of equilibrium. In this way

$$q^1 > 0. \tag{20}$$

Consider the model $\left(v^1, \frac{q^2}{q^1}, C^{12}, C^{21}\right)$. Taking $\eta^i = \frac{K^i}{L^i}$ $(i = 1,2)$ we obtain

$$\max_{x \ge 0} \frac{W^1(x, p)}{[q, x]} = \frac{1}{q^1} \max_{\eta > 0} \frac{v^1 \eta + \min\left(\frac{\eta}{C^{11}}, \frac{1}{C^{21}}\right)}{\eta + \frac{q^2}{q^1}}, \tag{21}$$

As is shown in Lemma 1, the last maximum is reached on the assets

$$\bar{\eta}^1 = \begin{cases} \chi^1, & if \quad q^2 < q^1 u^1, \\ +\infty, & if \quad q^2 \ge q^1 u^1. \end{cases} \tag{22}$$

Now let's consider the model $\left(v^2, \frac{q^2}{q^1}, \frac{1}{b} C^{12}, \frac{1}{b} C^{22}\right)$.

Making the similar consideration as above, we get

$$\max_{x \ge 0} \frac{W^2(x, p)}{[q, x]} = \frac{1}{q^1} \max_{\eta > 0} \frac{v^2 \eta + b \min\left(\frac{\eta}{C^{12}}, \frac{1}{C^{22}}\right)}{\eta + \frac{q^2}{q^1}}, \tag{23}$$

and

$$\bar{\eta}^2 = \begin{cases} \chi^2, & if \quad q^2 < q^1 u^2, \\ +\infty, & if \quad q^2 \ge q^1 u^2. \end{cases} \tag{24}$$

Let $v^1 = v^2 = 0$. If $q^1 > 0$, then the equilibrium

$$\bar{\eta}^i = x^i, \; x^i = \frac{\lambda^i}{q^1 x^i + q^2}(x^i, 1) \qquad (i = 1,2). \tag{25}$$

Since is always true the user tasks (21) and (23) may have solutions (22) and (24) respectively, it becomes possible to describe the state of equilibrium of model M . The following numbers $\chi,\ \mu,\ \chi^i,\ u^i\ (i=1,2)$ used below are defined by (16).

Suppose $E=\left(q,x^1,x^2\right)$ is an equilibrium state of the model M, $q=\left(q^1,q^2\right)$ are equilibrium prices. We say that this equilibrium is of type E^1 if $q^2=0$, of type E^2, if $u^2\le\frac{q^2}{q^1}<u^1$, of type E^3, if $u^1\le\frac{q^2}{q^1}<u^2$ and of type E^4, if $\frac{q^2}{q^1}<\min(u^1,u^2)$.

Theorem 1. The model M may have only E^1, E^2, E^3 and E^4 types of equilibrium. And the equilibrium E^1 where

$$x^1=\frac{\lambda^1}{q^1}\left(1,\frac{1}{\chi^1}\right),\ x^2=\frac{\lambda^2}{q^1}\left(1,\frac{1}{\chi^2}\right),\ q^1=\frac{1}{\chi L}\left(\lambda^1+\lambda^2\right),\quad q^2=0$$

is realized if and only if

$$\chi=\chi^1=\chi^2 \text{ and } \mu>0;$$

The equilibrium E^2 where

$$x^1=\frac{\lambda^1}{q^1\chi^1+q^2}(\chi^1,1),\quad x^2=\frac{\lambda^2}{q^1}(1,0),$$

$$q^1=\frac{\lambda^2}{(\chi-\chi^1)L},\quad q^2=\frac{1}{L}\left(\lambda^1-\frac{\lambda^2\chi^1}{\chi-\chi^1}\right)$$

is realized if and only if

$$\chi>\chi^1 \quad \text{и} \quad \frac{\chi^1+u^2}{\chi-\chi^1}\le\mu<\frac{\chi^1+u^1}{\chi-\chi^1};$$

The equilibrium E^3, where

$$x^1=\frac{\lambda^1}{q^1\chi^1+q^2}(1,0),\quad x^2=\frac{\lambda^2}{q^1\chi^2+q^2}(\chi^2,1),$$

$$q^1=\frac{\lambda^1}{\left(\chi-\chi^2\right)L},\quad q^2=\frac{1}{L}\left(\lambda^2-\frac{\lambda^1\chi^2}{\chi-\chi^2}\right).$$

is realized if and only if

$$\chi>\chi^2 \quad \text{and} \quad \frac{\chi-\chi^2}{\chi^2+u^2}<\mu\le\frac{\chi-\chi^2}{\chi^2+u^1};$$

The equilibrium E^4, where

$$x^1 = \frac{\lambda^1}{q^1\chi^1 + q^2}(\chi^1,1), \quad x^2 = \frac{\lambda^2}{q^1\chi^2 + q^2}(\chi^2,1),$$

$$q^1 = \frac{1}{L}\left(\frac{\lambda^1}{\chi - \chi^2} - \frac{\lambda^2}{\chi^1 - \chi}\right), \; q^2 = \frac{1}{L}\left(\frac{\lambda^2\chi^1}{\chi^1 - \chi} - \frac{\lambda^1\chi^2}{\chi - \chi^2}\right)$$

is realized if and only if

$$\chi^2 < \chi < \chi^1 \text{ and } \frac{\chi - \chi^2}{\chi^1 - \chi}\frac{\chi^1 + \min(u^1,u^2)}{\chi^2 + \min(u^1,u^2)} < \mu < \frac{\chi - \chi^2}{\chi^1 - \chi}\frac{\chi^1}{\chi^2} \tag{26}$$

or

$$\chi^1 < \chi < \chi^2 \text{ and } \frac{\chi^2 - \chi}{\chi - \chi^1} < \mu < \frac{\chi^2 - \chi}{\chi - \chi^1}\frac{\chi^1 + \min(u^1u^2)}{\chi^2 + \min(u^1,u^2)}. \tag{27}$$

Note 1. More interesting the case is when $K^i \neq 0, L^i \neq 0, q^i \neq 0 \; (i = 1,2)$ takes place in the equilibrium. This case is possible only under the condition (26) or (27) (see Fig. 1, Fig.2).

On the Fig.1 (correspondingly Fig.2) the set of points (μ, χ) is given, by which the equilibrium E^2, E^3, E^4 is realized for the case $\chi^2 < \chi^1$ ($\chi^1 < \chi^2$).

Note 2. If the pair (μ, χ) does not belong to one of these sets given in Figure 1 and Fifure 2 then in the corresponding model M there exists only semi-equilibrium, i.e. a such vector of prices q may be found that only the inequality $x^1 + x^2 \leq X$ is fulfilled and $x^1 + x^2 \neq X$ for the vectors x^1, x^2 that are solutions of the

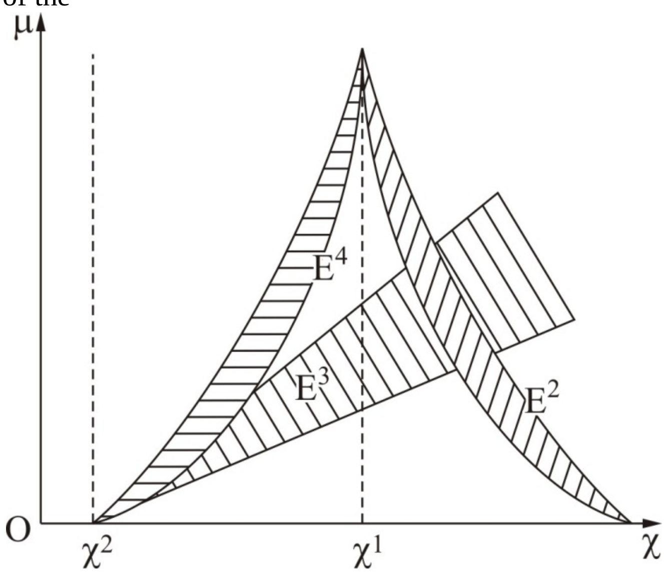

Figure 1.

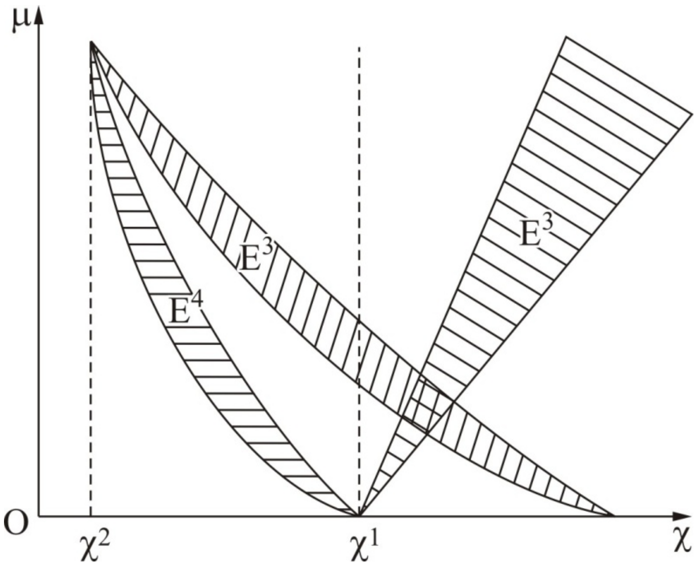

Figure 2

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О состоянии равновесия в двухсекторной модели экономической динамики

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РЕЗЮМЕ

Рассматривается двухсекторная модель экономической динамики, производящих соответственно средства производства и предметы потребления. Исследуется состояние равновесия модели с применением аппарата многозначных отображений. Определяются состояния равновесия модели.

Ключевые слова: равновесие, эффективные траектории, супер линейные отображения, производственные функции.